Lecture 6. Notes on Linear Algebra. Perceptron

COMP90051 Statistical Machine Learning

Semester 2, 2017
Lecturer: Andrey Kan

THE UNIVERSITY OF MELBOURNE

This lecture

- Notes on linear algebra
  - Vectors and dot products
  - Hyperplanes and vector normals
- Perceptron
  - Introduction to Artificial Neural Networks
  - The perceptron model
  - Stochastic gradient descent

Notes on Linear Algebra

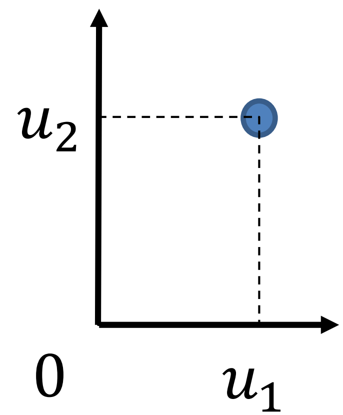
Link between geometric and algebraic interpretation of ML methods

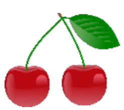
What are vectors?

Suppose $\boldsymbol{u} = [u_1, u_2]'$. What does $\boldsymbol{u}$ really represent?

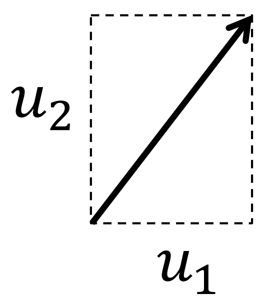
Ordered set of numbers $\{u_1, u_2\}$

Cartesian coordinates of a point

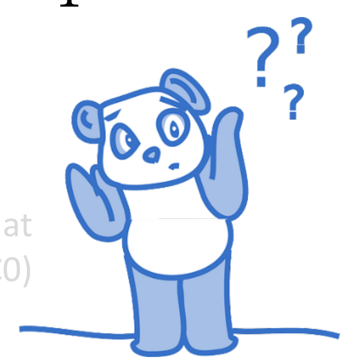




A direction



art: OpenClipartVectors at pixabay.com (CC0)



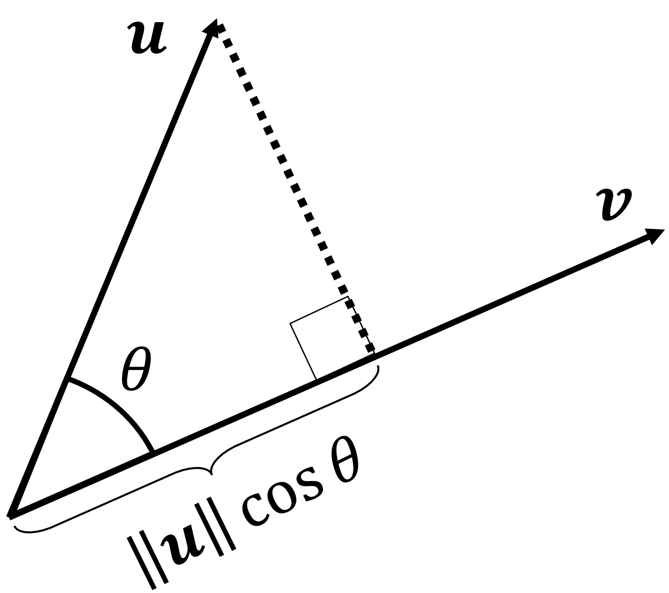
Dot product: algebraic definition

- Given two m-dimensional vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, their dot product is $\boldsymbol{u} \cdot \boldsymbol{v} \equiv \boldsymbol{u}'\boldsymbol{v} \equiv \sum_{i=1}^{m} u_i v_i$
  - E.g., weighted sum of terms is a dot product $\boldsymbol{x}'\boldsymbol{w}$
- Verify that if k is a constant and $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are vectors of the same size then

$$(k\boldsymbol{a})'\boldsymbol{b} = k(\boldsymbol{a}'\boldsymbol{b}) = \boldsymbol{a}'(k\boldsymbol{b})$$
$$\boldsymbol{a}'(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}'\boldsymbol{b} + \boldsymbol{a}'\boldsymbol{c}$$

Dot product: geometric definition

- Given two m-dimensional vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, their dot product is $\boldsymbol{u}\cdot\boldsymbol{v} \equiv \boldsymbol{u}'\boldsymbol{v} \equiv \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$
  - Here $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ are *L2 norms* (i.e., *Euclidean lengths*) for vectors $\boldsymbol{u}$ and $\boldsymbol{v}$
  - and θ is the angle between vectors



The *scalar projection* of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is given by

$$u_{\boldsymbol{v}} = \|\boldsymbol{u}\|\cos\theta$$

Thus dot product is

$$\boldsymbol{u}'\boldsymbol{v} = u_{\boldsymbol{v}}\|\boldsymbol{v}\| = v_{\boldsymbol{u}}\|\boldsymbol{u}\|$$

Equivalence of definitions

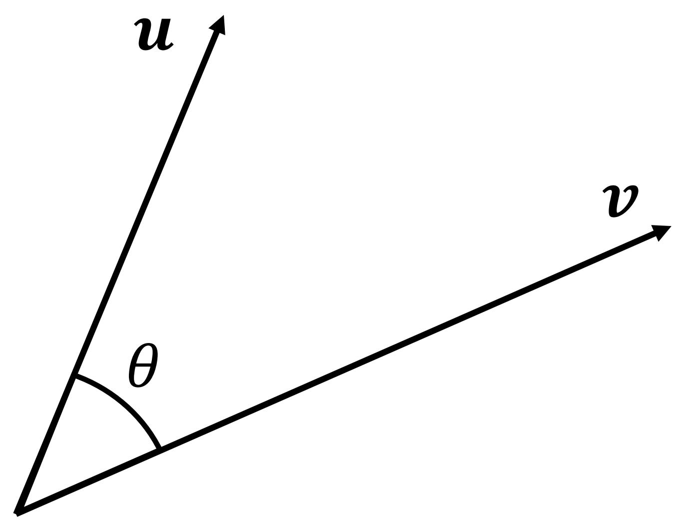
- Lemma: The algebraic and geometric definitions are identical
- Proof sketch:
  * Express the vectors using the standard vector basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ in $\boldsymbol{R}^m$, $\boldsymbol{u} = \sum_{i=1}^{m} u_i \boldsymbol{e}_i$, and $\boldsymbol{v} = \sum_{i=1}^{m} v_i \boldsymbol{e}_i$
  * Vectors $\boldsymbol{e}_i$ are an orthonormal basic, they have unit length and orthogonal to each other, so $\boldsymbol{e}_i' \boldsymbol{e}_i = 1$ and $\boldsymbol{e}_i' \boldsymbol{e}_j = 0$ for $i \neq j$

$$\boldsymbol{u}'\boldsymbol{v} = \boldsymbol{u}' \sum_{i=1}^{m} v_i \boldsymbol{e}_i = \sum_{i=1}^{m} v_i (\boldsymbol{u}' \boldsymbol{e}_i)$$

$$= \sum_{i=1}^{m} v_i (\|\boldsymbol{u}\| \|\boldsymbol{e}_i\| \cos\theta_i) = \sum_{i=1}^{m} v_i u_i$$

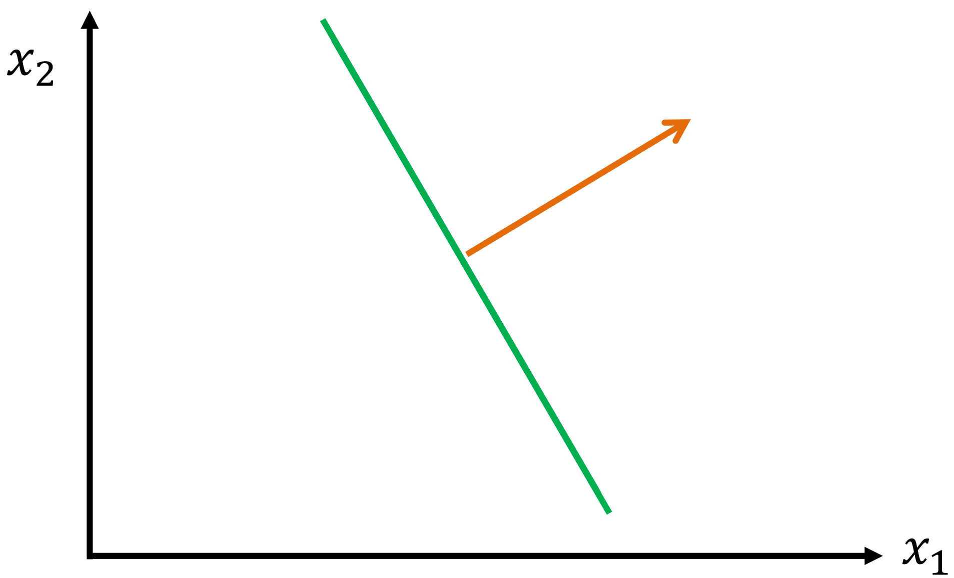
Geometric properties of the dot product

- If the two vectors are orthogonal then $\boldsymbol{u}'\boldsymbol{v} = 0$
- If the two vectors are parallel then $\boldsymbol{u}'\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|$, if they are anti-parallel then $\boldsymbol{u}'\boldsymbol{v} = -\|\boldsymbol{u}\|\|\boldsymbol{v}\|$
- $\boldsymbol{u}'\boldsymbol{u} = \|\boldsymbol{u}\|^2$, so $\|\boldsymbol{u}\| = \sqrt{u_1^2 + \cdots + u_m^2}$ defines the Euclidean vector length



Hyperplanes and normal vectors

- A *hyperplane* defined by parameters $\boldsymbol{w}$ and b is a set of points $\boldsymbol{x}$ that satisfy $\boldsymbol{x}'\boldsymbol{w} + b = 0$
- In 2D, a hyperplane is a line: a line is a set of points that satisfy $w_1x_1 + w_2x_2 + b = 0$



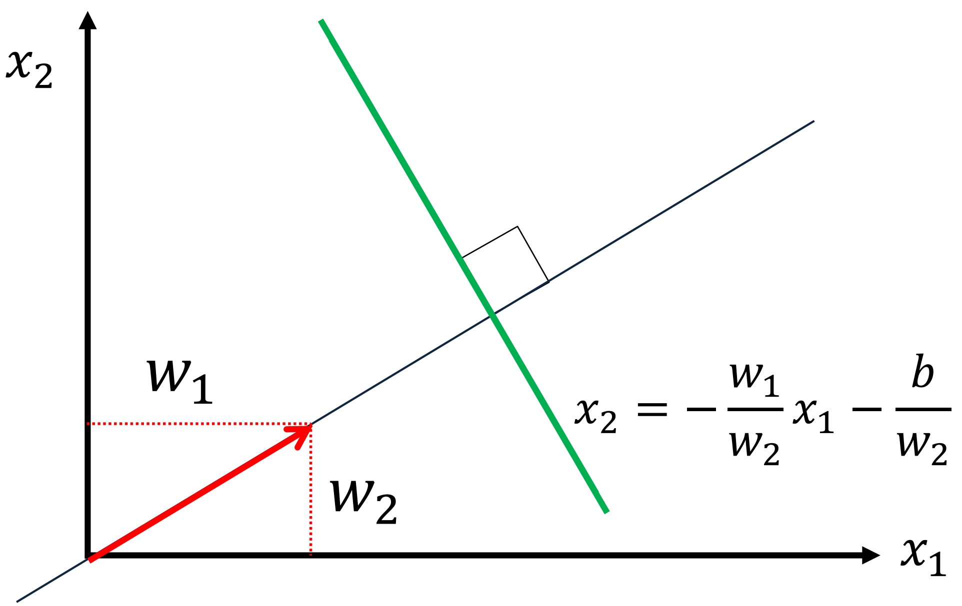
A *normal vector* for a hyperplane is a vector perpendicular to that hyperplane

Hyperplanes and normal vectors

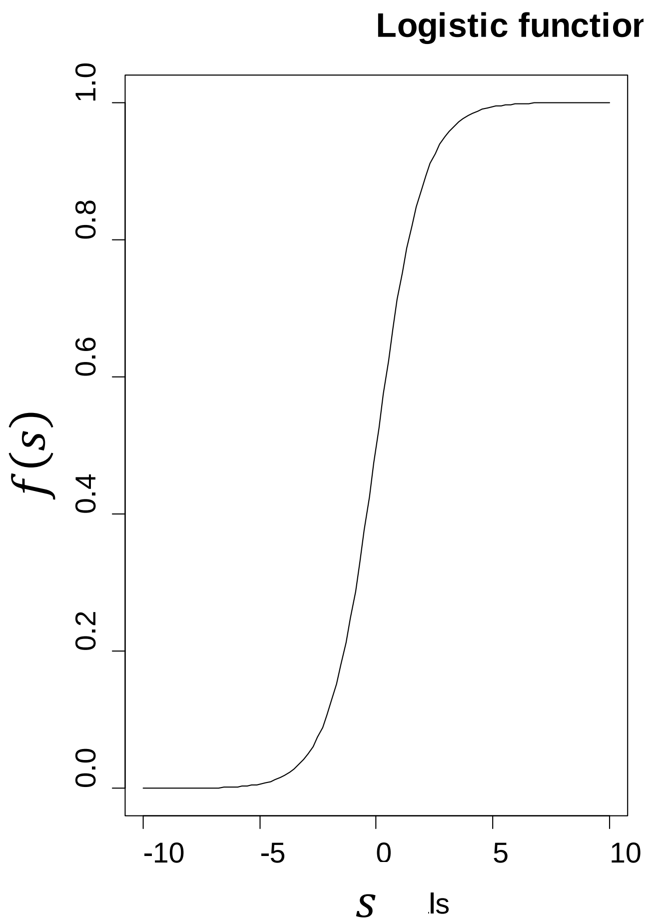
- Consider a hyperplane defined by parameters $\boldsymbol{w}$ and b. Note that $\boldsymbol{w}$ is itself a vector
- Lemma: Vector $\boldsymbol{w}$ is a normal vector to the hyperplane
- Proof sketch:
  * Choose any two points $\boldsymbol{u}$ and $\boldsymbol{v}$ on the hyperplane. Note that vector $(\boldsymbol{u} - \boldsymbol{v})$ lies on the hyperplane
  * Consider dot product $(\boldsymbol{u} - \boldsymbol{v})'\boldsymbol{w} = \boldsymbol{u}'\boldsymbol{w} - \boldsymbol{v}'\boldsymbol{w}$
  $$= (\boldsymbol{u}'\boldsymbol{w} + b) - (\boldsymbol{v}'\boldsymbol{w} + b) = 0$$
  * Thus $(\boldsymbol{u} - \boldsymbol{v})$ lies on the hyperplane, but is perpendicular to $\boldsymbol{w}$, and so $\boldsymbol{w}$ is a vector normal

Example in 2D

- Consider a line defined by w_1, w_2 and b
- Vector $\boldsymbol{w} = [w_1, w_2]'$ is a normal vector



Is logistic regression a linear method?



Logistic regression is a linear classifier

- Logistic regression model:

$$P(\mathcal{Y} = 1|\boldsymbol{x}) = \frac{1}{1 + \exp(-\boldsymbol{x}'\boldsymbol{w})}$$

- Classification rule:

if $\left(P(\mathcal{Y} = 1|\boldsymbol{x}) > \frac{1}{2}\right)$ then class "1", else class "0"

- Decision boundary:

$$\frac{1}{1 + \exp(-\boldsymbol{x}'\boldsymbol{w})} = \frac{1}{2}$$

$$\exp(-\boldsymbol{x}'\boldsymbol{w}) = 1$$

$$\boldsymbol{x}'\boldsymbol{w} = 0$$

The Perceptron Model

A building block for artificial neural networks, yet another linear classifier

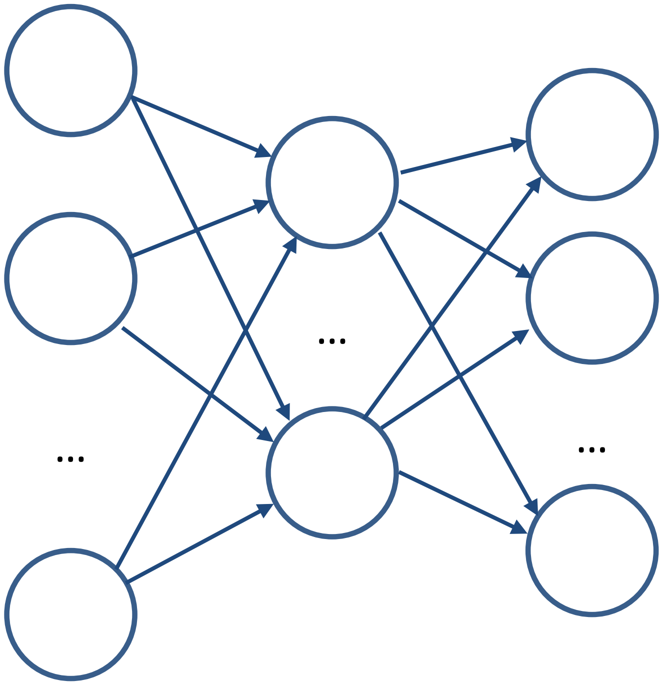
Biological inspiration

- Humans perform well at many tasks that matter
- Originally neural networks were an attempt to mimic the human brain

photo: Alvesgaspar, Wikimedia Commons, CC3

Artificial neural network

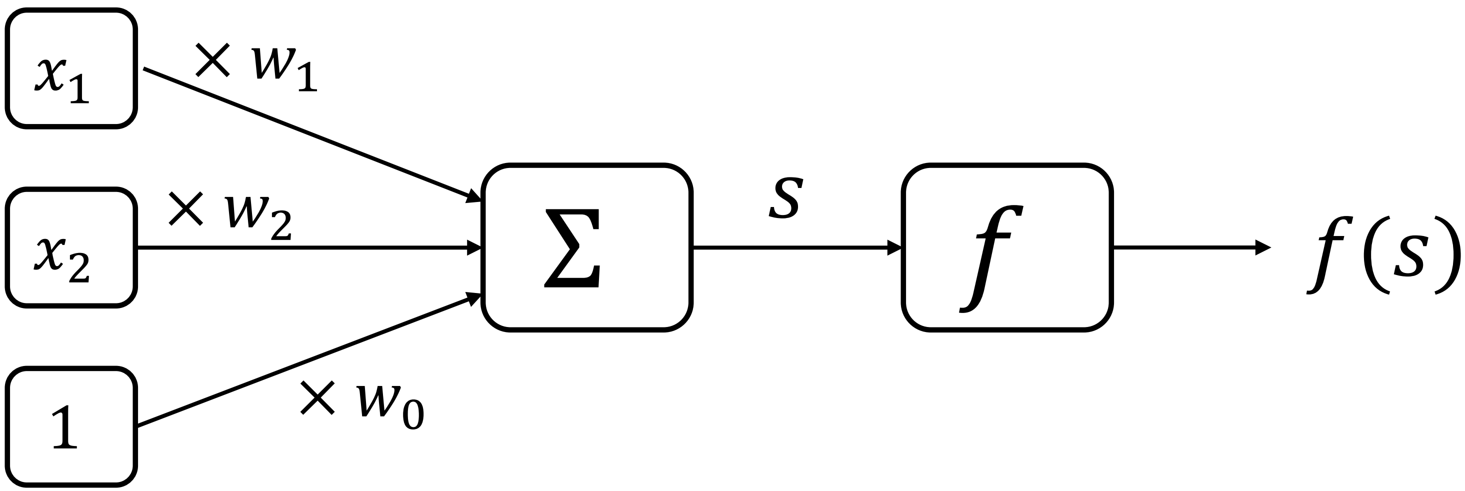
- As a *crude approximation*, the human brain can be thought as a mesh of interconnected processing nodes (neurons) that relay electrical signals
- Artificial neural network is a network of processing elements
- Each element converts inputs to output
- The output is a function (called *activation function*) of a weighted sum of inputs



Outline

- In order to use an ANN we need (a) to design network topology and (b) adjust weights to given data
  - In this course, we will exclusively focus on task (b) for a particular class of networks called feed forward networks
- Training an ANN means adjusting weights for training data given a pre-defined network topology
- We will come back to ANNs and discuss ANN training in the next lecture
- Right now we will turn our attention to an individual network element because it is an interesting model in itself

Perceptron model



Compare this model to logistic regression

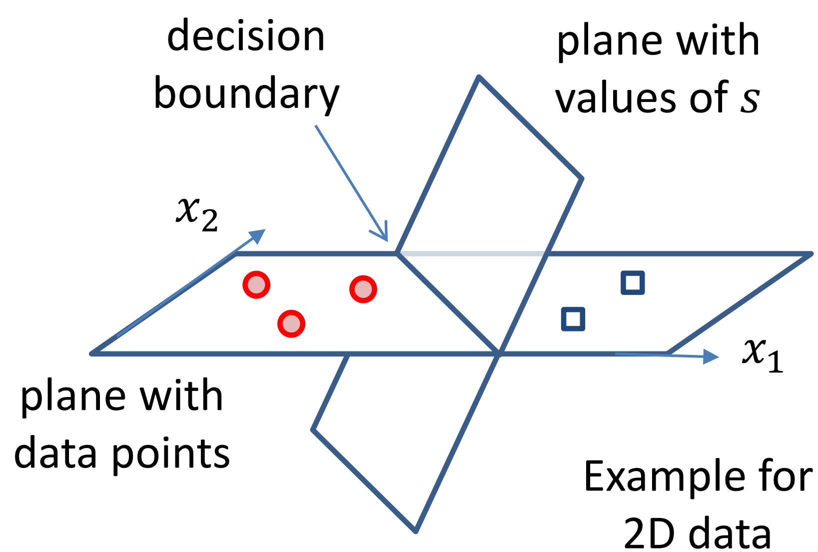
- x_1, x_2 – inputs
- w_1, w_2 – synaptic weights
- w_0 – bias weight
- f – activation function

Perceptron is a linear binary classifier

Perceptron is a binary classifier:

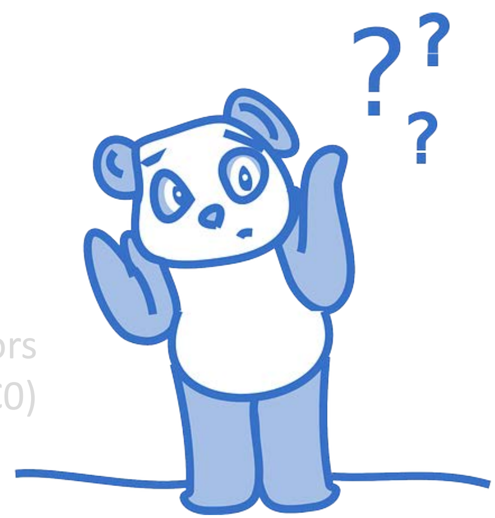
Predict class A if $s \geq 0$
Predict class B if $s < 0$
where $s = \sum_{i=0}^{m} x_i w_i$

Perceptron is a linear classifier: s is a linear function of inputs, and the decision boundary is linear



Exercise: find weights of a perceptron capable of perfect classification of the following dataset

x_1	x_2	y
0	0	Class B
0	1	Class B
1	0	Class B
1	1	Class A



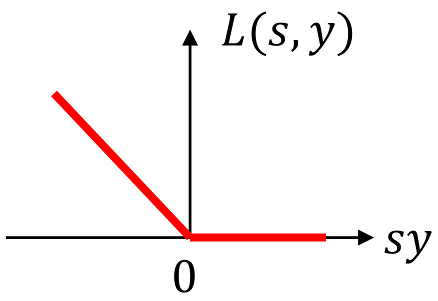
art: OpenClipartVectors at pixabay.com (CC0)

Loss function for perceptron

- Recall that "training" means finding weights that minimise some loss. Therefore, we proceed with considering the loss function for perceptron
- Our task is binary classification. Let's arbitrarily encode one class as $+1$ and the other as -1. So each training example is now $\{\boldsymbol{x}, y\}$, where y is either $+1$ or -1
- Recall that, in a perceptron, $s = \sum_{i=0}^{m} x_i w_i$, and the sign of s determines the predicted class: $+1$ if $s > 0$, and -1 if $s < 0$
- Consider a single training example. If y and s have the same sign then the example is classified correctly. If y and s have different signs, the example is misclassified

Loss function for perceptron

- Consider a single training example. If y and s have the same sign then the example is classified correctly. If y and s have different signs, the example is misclassified
- The perceptron uses a loss function in which there is no penalty for correctly classified examples, while the penalty (loss) is equal to s for misclassified examples*
- Formally:
  - $L(s,y)=0$ if both s,y have the same sign
  - $L(s,y)=|s|$ if both s,y have different signs



- This can be re-written as $L(s,y)=\max(0,-sy)$

* This is similar, but not identical to another widely used loss function called ***hinge loss***

Stochastic gradient descent

- Split all training examples in B batches
- Choose initial $\boldsymbol{\theta}^{(1)}$
- For i from 1 to T
- For j from 1 to B
- Do gradient descent update using data from batch j

Iterations over the entire dataset are called *epochs*

- Advantage of such an approach: computational feasibility for large datasets

Perceptron training algorithm

Choose initial guess $\boldsymbol{w}^{(0)}, k = 0$

For i from 1 to T (epochs)

For j from 1 to N (training examples)

Consider example $\{\boldsymbol{x}_j, y_j\}$

Update*: $\boldsymbol{w}^{(k++)} = \boldsymbol{w}^{(k)} - \eta \boldsymbol{\nabla} L(\boldsymbol{w}^{(k)})$

$$L(\boldsymbol{w}) = \max(0, -sy)$$

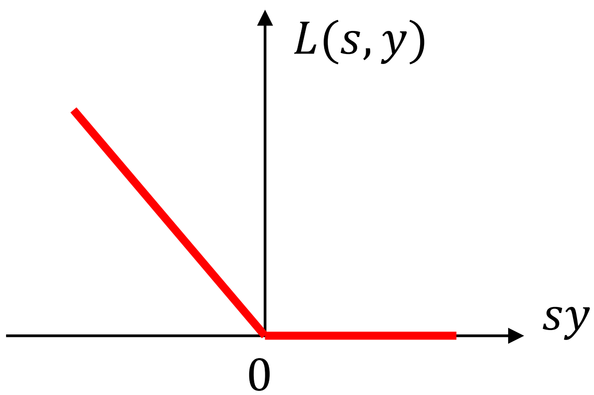
$$s = \sum_{i=0}^{m} x_i w_i$$

η is learning rate

*There is no derivative when $s = 0$, but this case is handled explicitly in the algorithm, see next slides

Perceptron training rule

- We have $\frac{\partial L}{\partial w_i} = 0$ when $sy > 0$
  - We don't need to do update when an example is correctly classified
- We have $\frac{\partial L}{\partial w_i} = -x_i$ when $y = 1$ and $s < 0$
- We have $\frac{\partial L}{\partial w_i} = x_i$ when $y = -1$ and $s > 0$
- $s = \sum_{i=0}^{m} x_i w_i$



Perceptron training algorithm

When classified correctly, weights are unchanged

When misclassified: $\boldsymbol{w}^{(k+1)} = -\eta(\pm\boldsymbol{x})$
($\eta > 0$ is called *learning rate)*

<u>If $y = 1$, but $s < 0$</u>
$w_i \leftarrow w_i + \eta x_i$
$w_0 \leftarrow w_0 + \eta$

<u>If $y = -1$, but $s \geq 0$</u>
$w_i \leftarrow w_i - \eta x_i$
$w_0 \leftarrow w_0 - \eta$

<u>Convergence Theorem</u>: if the training data is linearly separable, the algorithm is guaranteed to converge to a solution. That is, there exist a finite K such that $L(\boldsymbol{w}^K) = 0$

Perceptron convergence theorem

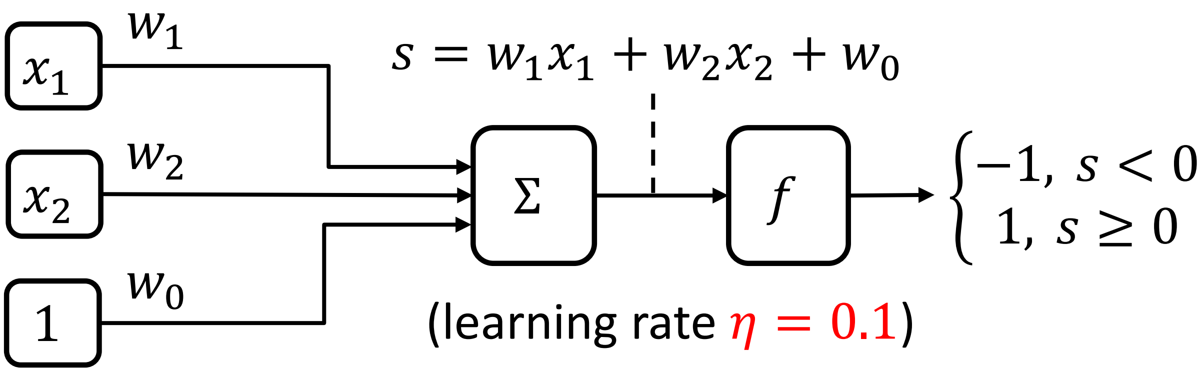
- Assumptions
  - Linear separability: There exists $\boldsymbol{w}^*$ so that $y_i(\boldsymbol{w}^*)'\boldsymbol{x}_i \geq \gamma$ for all training data $i = 1, \dots, N$ and some positive γ.
  - Bounded data: $\|\boldsymbol{x}_i\| \leq R$ for $i = 1, \dots, N$ and some finite R.
- Proof sketch
  - Establish that $(\boldsymbol{w}^*)'\boldsymbol{w}^{(k)} \geq (\boldsymbol{w}^*)'\boldsymbol{w}^{(k-1)} + \gamma$
  - It then follows that $(\boldsymbol{w}^*)'\boldsymbol{w}^{(k)} \geq k\gamma$
  - Establish that $\left\|\boldsymbol{w}^{(k)}\right\|^2 \leq kR^2$
  - Note that $\cos\left(\boldsymbol{w}^*, \boldsymbol{w}^{(k)}\right) = \frac{(\boldsymbol{w}^*)'\boldsymbol{w}^{(k)}}{\|\boldsymbol{w}^*\|\left\|\boldsymbol{w}^{(k)}\right\|}$
  - Use the fact that $\cos(\dots) \leq 1$
  - Arrive at $k \leq \frac{R^2\|\boldsymbol{w}^*\|^2}{\gamma}$

Pros and cons of perceptron learning

- If the data is linearly separable, the perceptron training algorithm will converge to a correct solution
  - There is a formal proof ← good!
  - It will converge to some solution (separating boundary), one of infinitely many possible ← bad!
- However, if the data is not linearly separable, the training will fail completely rather than give some approximate solution
  - Ugly ☹

Perceptron Learning Example

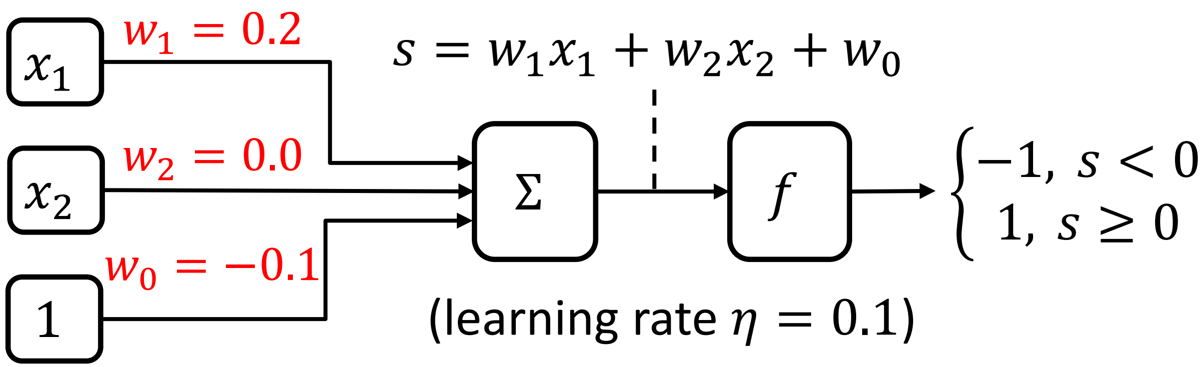
Basic setup

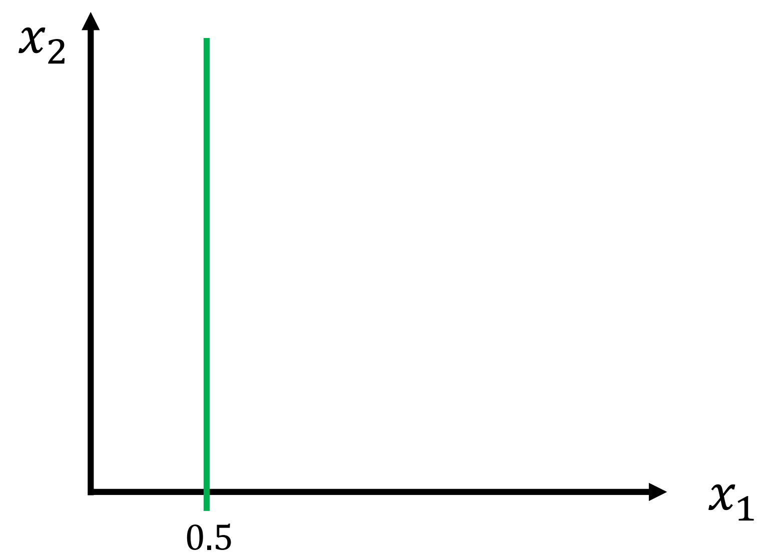


(learning rate $\eta = 0.1$)

Perceptron Learning Example

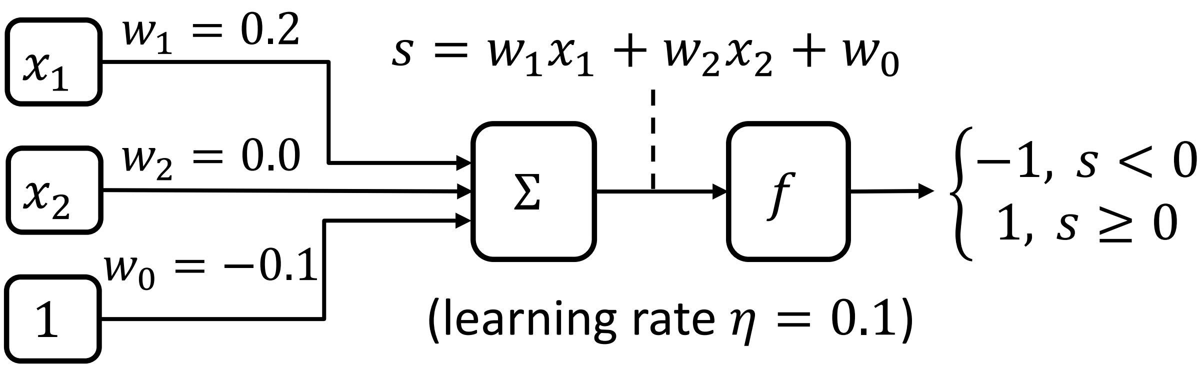
Start with random weights





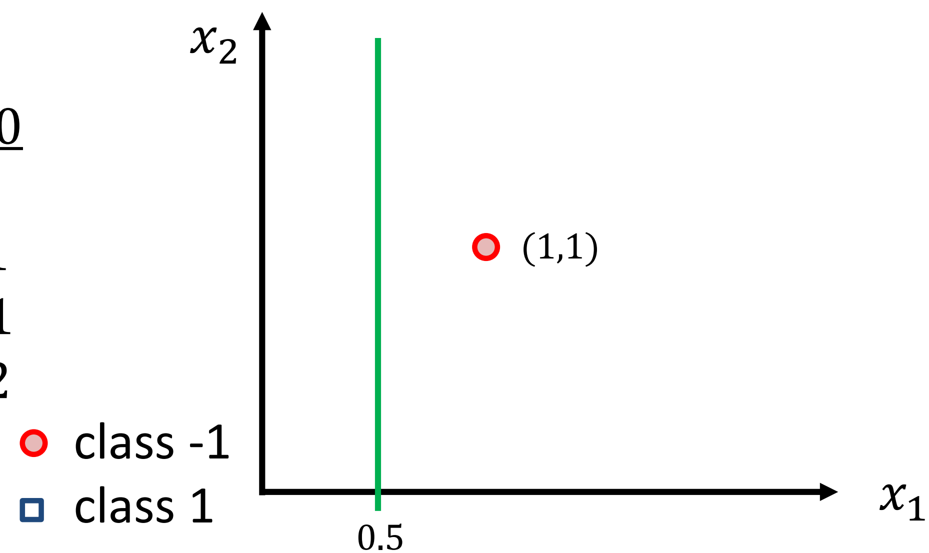
Perceptron Learning Example

Consider training example 1



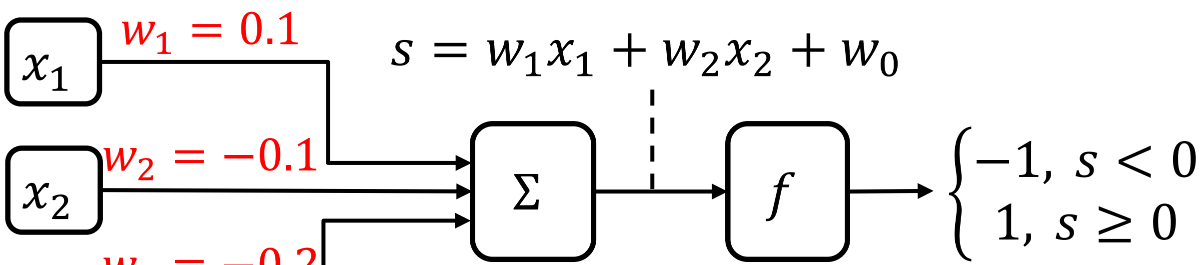
$\underline{0.2x_1 + 0.0x_2 - 0.1 > 0}$

$$\begin{aligned} w_1 &\leftarrow w_1 - \eta x_1 = 0.1 \\ w_2 &\leftarrow w_2 - \eta x_2 = -0.1 \\ w_0 &\leftarrow w_0 - \eta = -0.2 \end{aligned}$$

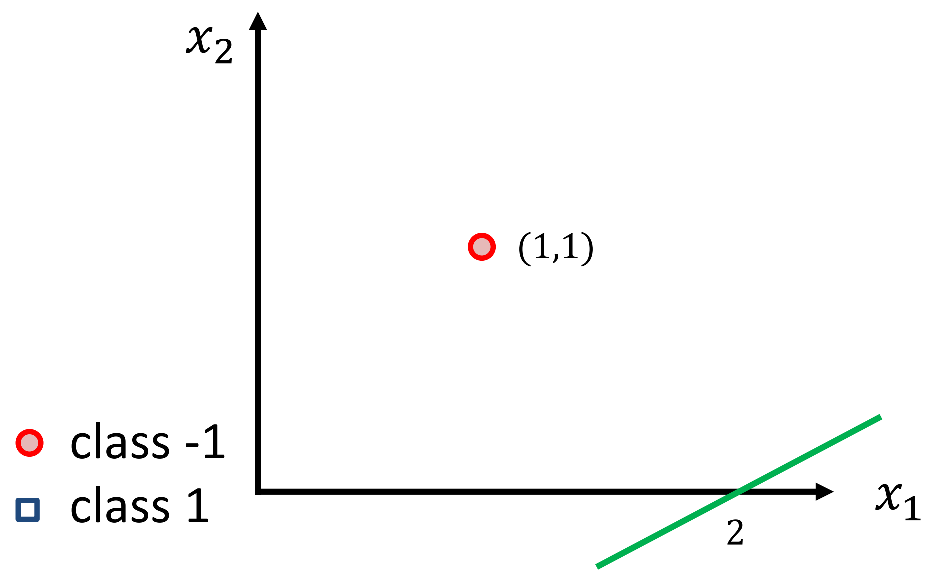


Perceptron Learning Example

<u>Update weights</u>

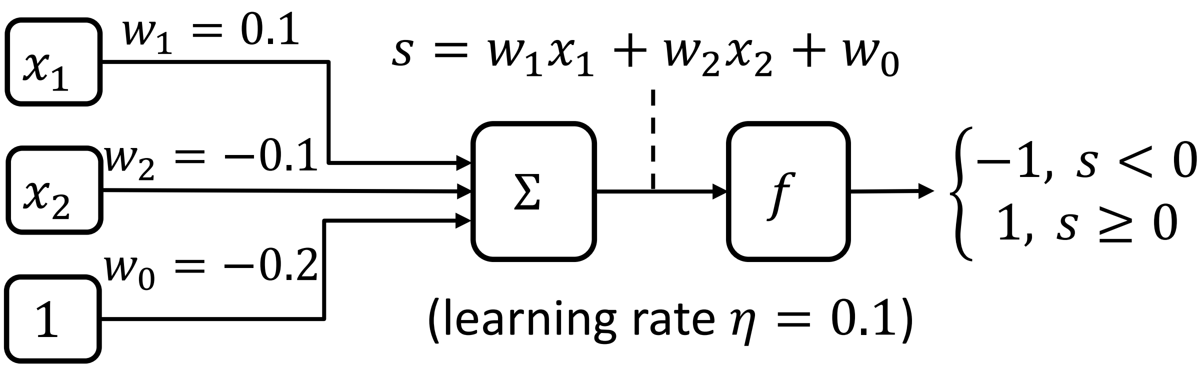


(learning rate $\eta = 0.1$)



Perceptron Learning Example

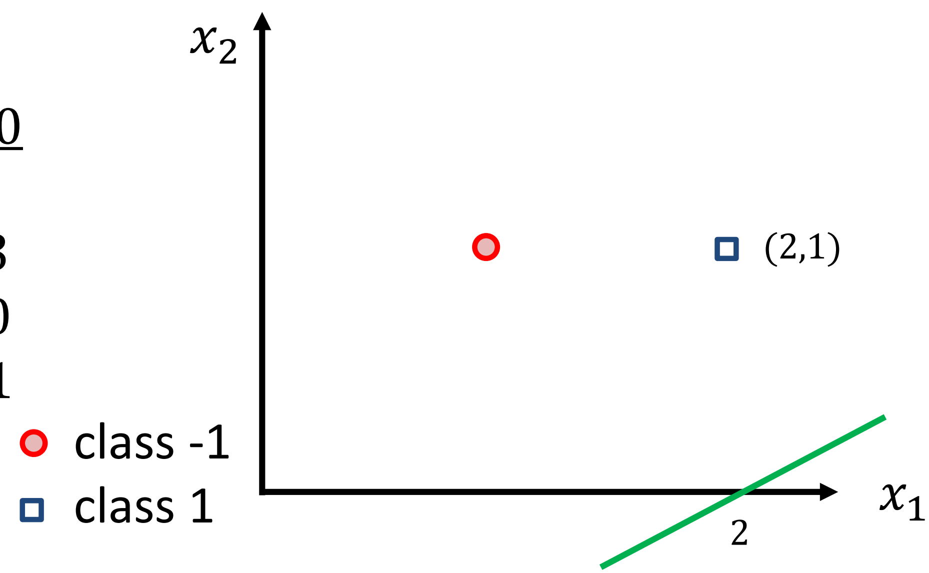
Consider training example 2



(learning rate $\eta = 0.1$)

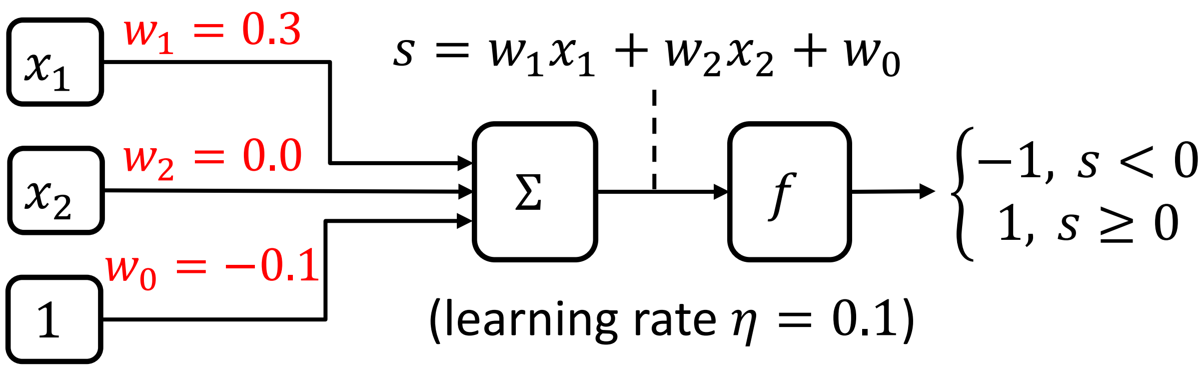
$0.1x_1 - 0.1x_2 - 0.2 < 0$

$$\begin{aligned} w_1 &\leftarrow w_1 + \eta x_1 = 0.3 \\ w_2 &\leftarrow w_2 + \eta x_2 = 0.0 \\ w_0 &\leftarrow w_0 + \eta = -0.1 \end{aligned}$$

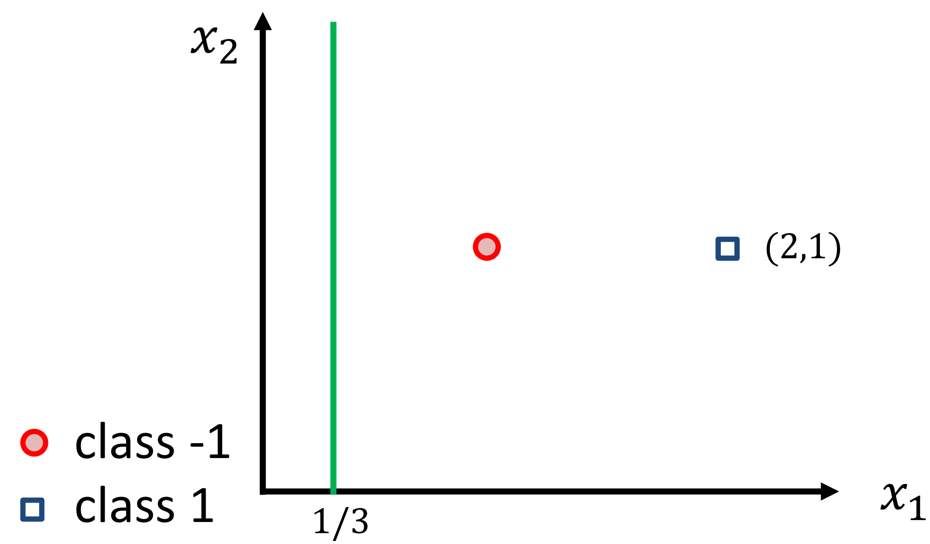


Perceptron Learning Example

Update weights

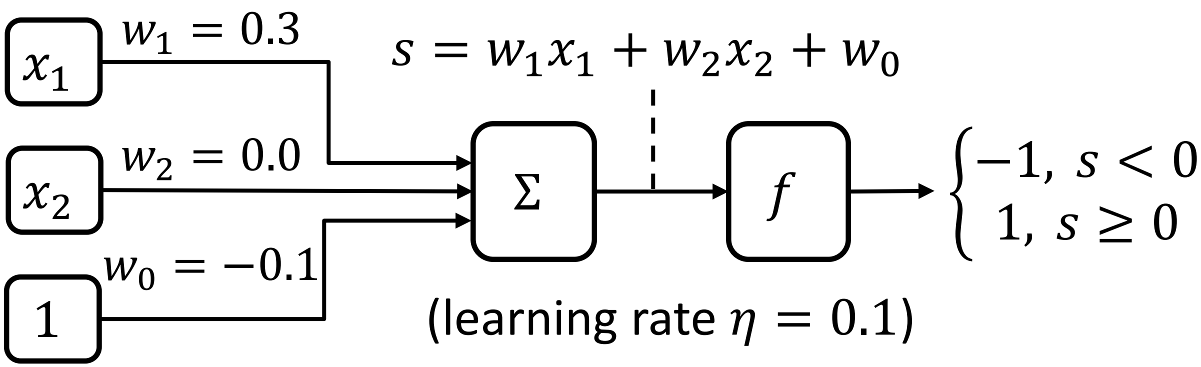


(learning rate $\eta = 0.1$)



Perceptron Learning Example

Further examples



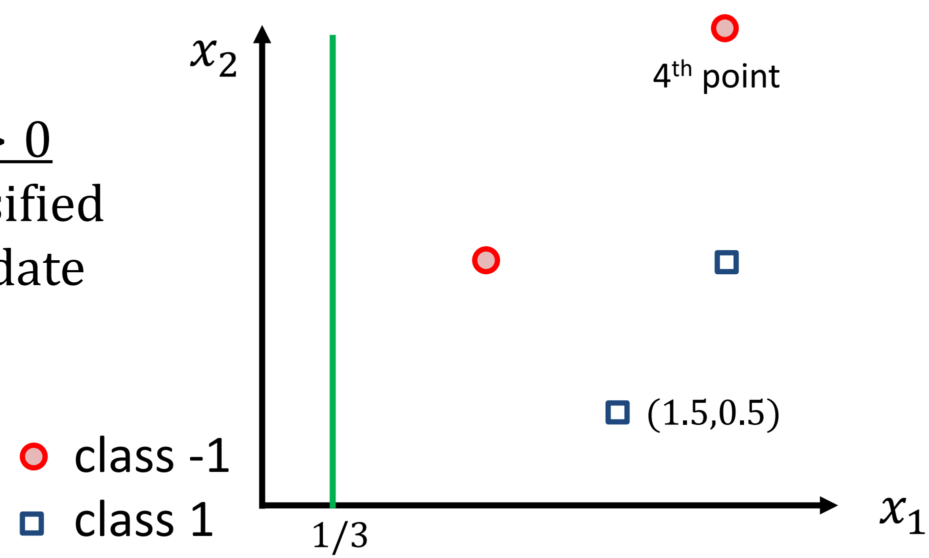
(learning rate $\eta = 0.1$)

$\underline{0.3x_1 - 0.0x_2 - 0.1 > 0}$

3rd point: correctly classified

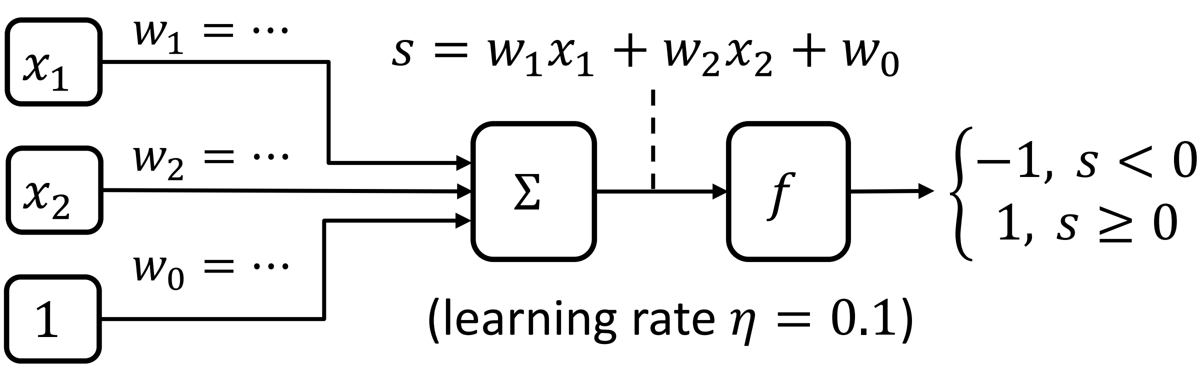
4th point: incorrect, update

etc.



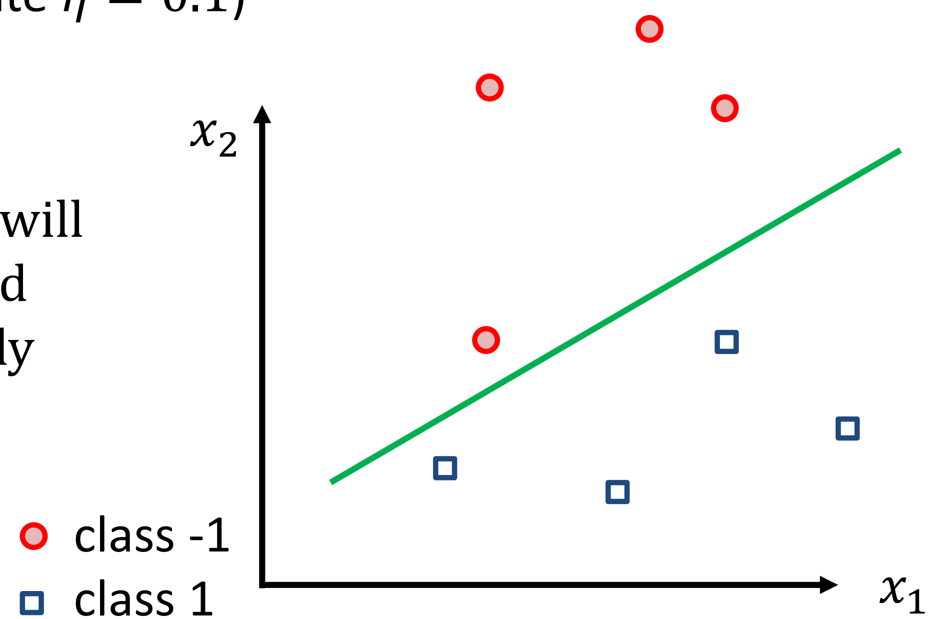
Perceptron Learning Example

<u>Further examples</u>



(learning rate $\eta = 0.1$)

Eventually, all the data will be correctly classified (provided it is linearly separable)



This lecture

- Notes on linear algebra
  - Vectors and dot products
  - Hyperplanes and vector normals
- Perceptron
  - Introduction to Artificial Neural Networks
  - The perceptron model
  - Training algorithm